Asymmetric Information and Rationalizability

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Abstract

We study how asymmetric information affects the set of rationalizable solutions in a linear setup where the outcome is determined by forecasts about this same outcome. The unique rational expectations equilibrium is also the unique rationalizable solution when the sensitivity of the outcome to agents' forecasts is less than one, provided that this sensitivity is common knowledge. Relaxing this common knowledge assumption, multiple rationalizable solutions arise when the proportion of agents who know the sensitivity is large, and the uninformed agents believe it is possible that the sensitivity is greater than one. Instability is equivalent to existence of some kind of sunspot equilibria.

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1 Introduction

The Rational Expectations Equilibrium (REE) supposes common knowledge (CK) of expectations. Following Guesnerie (1992), one can assess the relevance of this assumption by considering rationalizable outcomes. An outcome is rationalizable whenever it is consistent with CK of rationality and model. The REE is always a rationalizable outcome, but it is not necessarily the only rationalizable outcome since rationalizability entails no prior knowledge about expectations.

When the REE is the only rationalizable outcome, agents' expectations and the market outcome (prices, allocations) are uniquely determined by the fundamentals. This is no longer the case when there are multiple rationalizable outcomes. Expectations can then be heterogeneous and wrong[1] and the market outcomes are no longer pinned down by fundamentals. In this case, the REE is 'unstable.'

We address the stability issue in a linear coordination game with a continuum of players. In this class of games, stability is determined by the sensitivity of the actual aggregate outcome to agents' beliefs about it. Under CK of the sensitivity, Guesnerie (1992) shows that the REE is stable if and only if the sensitivity parameter is smaller than one. We relax the CK assumption and introduce asymmetric information about the sensitivity: Some agents are perfectly informed and the others have no private information about the sensitivity.

Suppose that the sensitivity differs across states of nature. Our main result is that asymmetric information about the sensitivity can lead to instability even in states of nature where the REE is the unique rationalizable outcome under CK of the sensitivity.

The intuition for this result hinges on a contagion-like argument across the different states of nature. Every agent wants to predict the outcome and, to achieve this goal, needs to expect others' behavior. But why should an informed agent care about the outcome in a state that does not occur? To predict the outcome in the true state, the informed agent needs to predict uninformed agents' behavior. To determine his own behavior, every uninformed agent must predict the outcome in every state since he does not know the true state. Everyone therefore takes account of all the states to predict the outcome. Both informed and uninformed agents are crucial for the contagion effect across states.

When there is a state of nature where the sensitivity is greater than one, this contagion-like argument implies that adding a small mass of uninformed agents to a world with perfectly informed agents is enough to lead to instability. Indeed, informed agents are unable to predict the outcome in the state where the sensitivity is greater than one. Therefore, uninformed agents cannot predict the outcome in this state, and instability spills over into other states through the forecasts of uninformed agents.

We show that instability under asymmetric information obtains if and only if there is a state where the sensitivity is greater than one and the proportion of informed agents is

[1] Dominitz and Manski (2007), Dominitz and Manski (2011) and Arrondel et al. (2012) provide recent empirical evidence about heterogeneous expectations.

high enough. The main implication of this result is that a higher proportion of informed agents can be 'destabilizing': 'More information' leads to instability.

Alternative assessments of REE instability refer to multiplicity of REE or existence of sunspot equilibria. We introduce a concept of sunspot equilibrium adapted to our static framework to discuss instability of the REE using the sunspot approach. We show equivalence between multiplicity of rationalizable outcomes and existence of a sunspot equilibrium.

In the main strand of the literature, asymmetric information in linear coordination games has been analyzed under CK of the sensitivity (Morris and Shin (2002), Angeletos and Pavan (2004), Angeletos and Pavan (2007), Hellwig (2005), Cornand and Heinemann (2010)). We depart from the literature by assuming that uncertainty bears on the sensitivity parameter.

Our equivalence between instability and existence of sunspot equilibrium is reminiscent of equivalence results found in dynamic models in Guesnerie (1993). Guesnerie (2011) discuss the links between various concepts based on CK ideas.

The paper is organized as follows. The benchmark setup is presented in Section 2. The case of complete information is briefly described in Section 3. In Section 4, the analysis is extended to the case of asymmetric information, and the main results are given. In Section 5, we consider extraneous uncertainty of the sunspot type.

2 The framework

We consider a stylized model with a beauty contest issue. There is a continuum of infinitesimal agents $i \in [0,1]$ who simultaneously form forecasts p_i^e about the 'price.' These forecasts then determine the actual price. The uncertainty about fundamentals is represented by Ω states of nature indexed by ω, $\omega = 1, \ldots, \Omega$. In state ω, the actual price is

$$p(\omega) = \phi(\omega) \int_0^1 p_i^e di + \eta(\omega). \tag{1}$$

Fundamentals in state ω are summarized by the pair $(\phi(\omega), \eta(\omega))$, where $\phi(\omega)$ measures the sensitivity of the actual price to forecasts and $\eta(\omega)$ is a scale factor. The reduced form used by Morris and Shin (2002) fits (1) with $\phi(\omega) = \phi \in (0,1)$. In the sequel, we also assume that the model exhibits strategic complementarity, i.e., $\phi(\omega) > 0$ for every ω. Our analysis would apply in the presence of strategic substitutability, i.e., $\phi(\omega) < 0$ for every ω, as in the agricultural model of Guesnerie (1992). However it does not extend to the case where the signs of the sensitivity to beliefs differ across states of nature.

Example 1. Muth model (Guesnerie, 1992). There is a continuum of farmers $i \in [0,1]$ who produce corn. Each farmer chooses his crop one period before observing the corn price. The cost of producing q units of corn is q^2/σ, with $\sigma > 0$. Farmer i expected profit is $p_i^e q - q^2/\sigma$ and thus his production is $q_i = \sigma p_i^e$. The actual price clears the market. The aggregate demand is $b - ap$. Aggregate supply equals aggregate demand when

$$\sigma \int p_i^e di = -ap + b,$$

which fits (1), with $\phi(\omega) = \phi = -\sigma/a < 0$. In this example, the sensitivity is the same in every state. The sensitivity would vary across states of nature with uncertain aggregate demand, e.g., $b(\omega) - a(\omega)p$ in state ω $(a(\omega), b(\omega) > 0)$.

Example 2. Lucas supply curve. There is a continuum of infinitesimal firms $i \in [0,1]$. Supply of firm i is $q_i = \sigma(p_i - p_i^e)$, where p_i stands for the price of its product and p_i^e represents its forecast about the aggregate price level. The aggregate price level in state ω is

$$p(\omega) \equiv \int_0^1 p_i(\omega) di.$$

The aggregate demand is $-a(\omega)p + b(\omega)$ in state ω when the aggregate price is p. In equilibrium, the aggregate price $p(\omega)$ satisfies

$$\int_0^1 \sigma(p_i(\omega) - p_i^e) di = -a(\omega)p(\omega) + b(\omega).$$

This fits (1), with $\phi(\omega) \equiv \sigma/(\sigma + a(\omega)) > 0$ and $\eta(\omega) \equiv b(\omega)/(\sigma + a(\omega))$.

3 Complete information

In (1), the individual price forecasts implicitly depend on agents' information. When it is commonly known that the state is ω, price forecasts are made conditionally on ω, i.e., $p_i^e = p_i^e(\omega)$ in (1). A rational expectations equilibrium (REE) is a price $p^*(\omega)$ solution to (1) when $p_i^e(\omega) = p^*(\omega)$ for all i. The REE is unique if and only if $\phi(\omega) \neq 1$.

The REE can be viewed as the Nash equilibrium of a strategic guessing game in which agent j chooses a forecast $p_j^e(\omega)$ which minimizes his forecast error $(p(\omega) - p_j^e(\omega))^2$, given that $p(\omega)$ is determined by (1). In this game, the best-response forecast of agent j to a profile $(p_i^e(\omega))$ of others' forecasts is

$$p_j^e(\omega) = \phi(\omega) \int_0^1 p_i^e di(\omega) + \eta(\omega). \tag{2}$$

Through this interpretation, every agent expects $p^*(\omega)$ because each one believes that all the others expect $p^*(\omega)$. This (second order) belief is justified by higher order beliefs such that all the agents believe that all the rest expect $p^*(\omega)$. The price $p^*(\omega)$ is the only one consistent with the common knowledge (CK) of every agent expecting it.

Following Guesnerie (1992), this interpretation suggests an assessment of the REE relying on a weaker assumption than CK of $p_i^e(\omega) = p^*(\omega)$ for all i. Assume instead that it is CK that the actual price $p(\omega)$ belongs to some set $P^0 = \left[p^0_{\text{inf}}, p^0_{\text{sup}}\right]$ which comprises $p^*(\omega)$. From this assumption, it is CK that $p^e_i(\omega) \in P^0$ for all i. Appealing to (1), all the agents can infer that the actual price will be in the set $P^1(\omega) = R_\omega(P^0)$ where the map R_ω is defined by

$$R_\omega\left(P\right) \equiv \left[\phi(\omega)P + \eta(\omega)\right] \cap P,$$

where P is any subset of prices. The actual price is determined by (1), provided that it is in P^0. Otherwise, it is the appropriate bound of P^0, either p^0_{inf} (if the price given by (1) is less than p^0_{inf}) or p^0_{sup} (if the price is greater than p^0_{inf}).

One defines a sequence of sets $P^\tau(\omega)$ along the same lines by $P^\tau(\omega) = R_\omega(P^{\tau-1}\left(\omega\right))$. It follows that if it is CK that $p(\omega) \in P^{\tau-1}(\omega)$, then it is CK that $p(\omega) \in P^\tau(\omega) = R_\omega(P^{\tau-1}(\omega))$. Then, the set of prices consistent with the common knowledge assumptions is the limit set

$$P^\infty\left(\omega\right) = \underset{\tau \geq 0}{\cap} P^\tau(\omega).$$

This limit set is properly defined since the sequence $P^\tau(\omega)$ is decreasing. The limit set is the set of rationalizable price forecasts of the guessing game (where forecasts are *a priori* restricted to P^0).

The equilibrium is 'stable' when $P^\infty\left(\omega\right) = \{p^*(\omega)\}$. Otherwise, the REE is 'unstable.' Every price in P^0 is rationalizable when the REE is unstable. The following condition for stability has been given by Guesnerie (1992):

Proposition 1. *The REE is stable if and only if* $\phi\left(\omega\right) < 1$.

This proposition provides a benchmark for our analysis of the asymmetric information case. Stability is obtained when the economic system is not too sensitive to forecasts in (1), or equivalently agents' forecasts are not too sensitive to others' forecasts in (2).

4 Asymmetric information

We now assume that there are only α ($0 \leq \alpha < 1$) 'informed' agents who observe ω before choosing their price forecasts. The $(1-\alpha)$ remaining agents have no information about the true state of nature at that time. These 'uninformed' agents have common prior beliefs: They all believe that state ω occurs with probability $\pi(\omega)$.

A REE is a vector of $(p^*(1), \ldots, p^*(\Omega))$ such that

$$p^*\left(\omega\right) = \phi\left(\omega\right)\left(\alpha p^*\left(\omega\right) + (1-\alpha)\sum_w \pi(w)p^*(w)\right) + \eta\left(\omega\right) \tag{3}$$

for any ω. The REE coincides with the Nash equilibrium of an amended guessing game in which agents try to minimize their forecast errors. This Bayesian game is as follows. First, the true state ω is observed only by the informed agents $i \in [0, \alpha]$. Then, all the agents simultaneously choose their forecasts. The strategy of agent i is a price forecast conditional on his information. If i is informed, his strategy is a vector of price forecasts $(p_i^e(1), ..., p_i^e(\Omega))$, where $p_i^e(\omega)$ is the price expected by i to arise in state ω. If i is uninformed, then his strategy merely consists of a single price forecast p_i^e independent of ω. The aggregate price forecast in state ω is therefore

$$\int_0^\alpha p_i^e(\omega)\,di + \int_\alpha^1 p_i^e di.$$

Finally, the actual price $p(\omega)$ is determined by the aggregate price forecast according to the map

$$p(\omega) = \phi(\omega)\left(\int_0^\alpha p_i^e(\omega)\,di + \int_\alpha^1 p_i^e di\right) + \eta(\omega). \tag{4}$$

Example 3. Muth model. The aggregate demand in state ω be $b(\omega) - a(\omega)p$. The expected profit of an informed farmer i is $p_i^e(\omega)q - q^2/\sigma$ and his supply is $q_i(\omega) = \sigma p_i^e(\omega)$. The expected profit of an uninformed farmer is $\sum_w \pi(w)p_i^e(w)q - q^2/\sigma$, so that his production is $q_i = \sigma \sum_w \pi(w)p_i^e(w)$. In equilibrium, the actual price $p(\omega)$ in state ω is such that

$$\sigma\left(\int_0^\alpha p_i^e(\omega)di + \int_\alpha^1 \sum_w \pi(w)p_i^e(w)di\right) = -a(\omega)p(\omega) + b(\omega).$$

Example 4. Lucas supply curve. If firm i is informed about the demand function, its supply is $q_i = \sigma(p_i(\omega) - p_i^e(\omega))$. If it is uninformed, its supply is $q_i = \sigma(p_i - \sum_w \pi(w)p_i^e(w))$. The aggregate price level is

$$p(\omega) \equiv \int_0^\alpha p_i^e(\omega)di + \int_\alpha^1 p_i di.$$

Therefore, in equilibrium,

$$\sigma p(\omega) - \sigma\left(\int_0^\alpha p_i^e(\omega)di + \int_\alpha^1 \sum_w \pi(w)p_i^e(w)di\right) = -a(\omega)p(\omega) + b(\omega).$$

Assume CK that the price *a priori* belongs to some interval P^0 which includes the equilibrium prices $p^*(\omega)$ for every ω. Every agent thus knows that all the other agents expect the price to be in P^0, and, consequently, each one understands that the aggregate price forecast is in P^0 in any state of nature. Hence, every agent concludes that the price in state ω belongs to the set $P^1(\omega) = R_\omega(P^0)$, which is included in P^0 and may coincide with P^0. When $P^1(\omega) \subsetneq P^0$, agents have succeeded in eliminating some price forecasts.

Iterating this process yields the CK restriction that the price in state ω is in some set $P^{\tau-1}(\omega)$ after $\tau - 1$ steps. At step τ, every agent knows that all the others expect the price in state ω to be in $P^{\tau-1}(\omega)$. Every agent understands that the price forecast in state ω of an informed agent is in $P^{\tau-1}(\omega)$, and that the price forecast of an uninformed agent is in $\sum_w \pi(w)P^{\tau-1}(w)$. All agents conclude that the price in state ω belongs to

$$P^\tau(\omega) = R_\omega\left(\alpha P^{\tau-1}(\omega) + (1-\alpha)\sum_w \pi(w)P^{\tau-1}(w)\right). \tag{5}$$

The relation (5) defines a sequence of intervals $(P^\tau(\omega), \tau \geq 0)$ for every ω. These sequences are decreasing and converge to limit sets $P^\infty(\omega)$. The REE is 'stable' whenever $P^\infty(\omega) = \{p^*(\omega)\}$ for every ω. Otherwise, it is 'unstable.'

As in Section 3, this definition has a game-theoretical counterpart in terms of rationalizable solutions (Bernheim (1984), Pearce (1984)). At step τ, if the strategy set is restricted to $\times_\omega P^{\tau-1}(\omega)$ for an informed agent, and to $\sum_w \pi(w) P^{\tau-1}(w)$ for an uninformed agent, then the best-response of agent i is a strategy in $\times_\omega P^\tau(\omega)$ when he is informed, and in $\sum_w \pi(w) P^\tau(w)$ when he is uninformed. The limit sets $P^\infty(\omega)$ are the rationalizable price forecasts of the 'guessing' game: $P^\infty(1) \times \cdots \times P^\infty(\Omega)$ is the set of rationalizable price forecasts of an informed agent, and $\sum_w \pi(w) P^\infty(w)$ is the set of rationalizable price forecasts of an uninformed one. Stability of the REE is equivalent to the uniqueness of the rationalizable price forecast, which then reduces to the REE prices.

4.1 (In)stability results

The following result presents the properties of the set of rationalizable prices when the REE is unstable.

Proposition 2. *Consider an unstable REE.*

1. *For every ω, $\{p^*(\omega)\} \subsetneq P^\infty(\omega)$: For every ω, the set $P^\infty(\omega)$ of rationalizable prices in state ω includes but differs from $\{p^*(\omega)\}$.*

2. *There is ω such that $p^\infty_{\inf}(\omega) = p^0_{\inf}$, and there is ω' (possibly different from ω) such that $p^\infty_{\sup}(\omega') = p^0_{\sup}$. In addition, for every ω such that $\alpha\phi(\omega) > 1$, $P^\infty(\omega) = P^0$.*

3. *For every ω such that $\phi(\omega) < 1$, $P^\infty(\omega) \subsetneq P^0$, and $P^\infty(\omega)$ decreases in P^0: If $P^0 \subsetneq \tilde{P}^0$, then the limit sets $P^\infty(\omega)$ and $\tilde{P}^\infty(\omega)$ associated with the initial restrictions P^0 and $\tilde{P}^0$ are such that $P^\infty(\omega) \subsetneq \tilde{P}^\infty(\omega)$.*

The first item of this Proposition is a formal statement of the 'contagion' property. It shows that no price $p^*(\omega)$ can be guessed in the case of instability, even in a state ω where $\phi(\omega) < 1$. Indeed, uninformed agents cannot select a single price forecast when the REE is unstable. This situation implies that, in every state, agents cannot settle upon the aggregate price forecast. Therefore, the actual price, which is determined by the aggregate price forecast, cannot be uniquely determined.

When the equilibrium is unstable, some 'coordination' volatility occurs in all the states at the outcome of the process of elimination of non-best response strategies. The magnitude of this volatility can be measured in state ω by the size of the interval $P^{\infty}(\omega)$ of rationalizable prices. Volatility is dampened when $P^{\infty}(\omega)$ is a narrow interval around the REE price $p^*(\omega)$. The second and the third items of Proposition 2 characterize how the residual volatility depends on economic fundamentals. They show that a low sensitivity to beliefs $\phi(\omega)$ plays a role reminiscent of that in the complete information case. A low sensitivity favors a narrow set $P^{\infty}(\omega)$ of rationalizable prices in state ω. In the contrary case, in a state where $\phi(\omega)$ is large enough, the iterative process (5) provides no additional information: $P^{\infty}(\omega) = P^0$. These two items also show how the magnitude of this volatility depends on the initial assumption made about the relevant prices: A narrower prior set P^0 yields a narrower set $P^{\infty}(\omega)$ of rationalizable prices at the outcome of (5).

Thus far, we have focused on the description of an unstable REE. The system (5) is a first-order linear recursive system. The REE is stable if and only if the spectral radius of the square matrix governing the dynamics (5) is less than 1. This yields the conditions for stability of the REE given in Proposition 3.

Proposition 3. *Assume that $\phi(\omega) > 0$ for any ω. Let $0 \le \alpha \le 1$.*

1. *If $\alpha\phi(\omega) > 1$ for some ω, then the REE is unstable.*
2. *If $\alpha\phi(\omega) < 1$ for every ω, then the REE is stable if and only if*

$$\sum_{w=1}^{\Omega} \pi(w) \frac{(1-\alpha)\phi(w)}{1-\alpha\phi(w)} < 1. \tag{6}$$

Point 1 in Proposition 3 states that the REE is stable in (5) only if $\alpha\phi(\omega) < 1$ for every ω. This inequality would also govern stability of the REE in state ω in a complete information setup involving α informed agents only. This fact suggests one should interpret this inequality by referring to a virtual restricted coordination problem which abstracts from the difficulties caused by uninformed agents. Namely, if informed agents know that the forecast of uninformed agents is fixed at $\bar{p}^* \equiv \sum_w \pi(w)p^*(w)$, then the actual price in state ω is

$$p(\omega) = \phi(\omega) \int_0^{\alpha} p_i^e(\omega)di + \tilde{\eta}(\omega)$$

where $\tilde{\eta}(\omega) = (1-\alpha)\phi(\omega)\bar{p}^* + \eta(\omega)$. This virtual restricted setup is formally equivalent to the complete information case discussed in the previous section (with a mass α of agents only). Hence, by Proposition 1, the REE is unstable when $\alpha\phi(\omega) > 1$. In this configuration, informed agents cannot correctly predict the price in state ω, even though, they know that uninformed agents expect the REE prices. It follows that in the true unrestricted setup, no agent (neither informed nor uninformed) succeeds in predicting the price in such a state.

Along the same lines, Point 2 in Proposition 3 can be interpreted as a stability condition of a virtual restricted problem which abstracts from the difficulties caused by the informed agents. Namely, if informed agents correctly guess the price, then the actual price is

$$p(\omega) = \phi(\omega)\left(\alpha p(\omega) + \int_{1-\alpha}^{1} \bar{p}_i^e di\right) + \eta(\omega),$$

and the actual average price is

$$\sum_{w=1}^{\Omega} \pi(w)\, p(w) = \left(\sum_{w=1}^{\Omega} \pi(w) \frac{\phi(w)}{1-\alpha\phi(w)}\right) \int_{1-\alpha}^{1} \bar{p}_i^e di + \hat{\eta}(\omega),$$

where

$$\hat{\eta}(\omega) = \sum_{w=1}^{\Omega} \pi(w) \frac{\eta(w)}{1-\alpha\phi(w)}.$$

Again this virtual restricted setup is formally equivalent to the complete information case with a mass $1-\alpha$ of agents. By Proposition 1, stability of this virtual setup is given by (6).

The following Corollary to Proposition 3 describes how stability is affected by the information structure.

Corollary 1. *Let $\phi(\omega) > 0$ for all ω. Let also $\alpha < 1$. Then, there is a unique threshold proportion α^*, $0 \leq \alpha^* \leq 1$, of informed agents such that stability of the REE is obtained if and only if $\alpha < \alpha^*$. In addition,*

1. *if $\phi(\omega) < 1$ for any ω, then $\alpha^* = 1$,*
2. *if there is ω with $\phi(\omega) > 1$ and if $\bar{\phi} = \sum \pi(w)\phi(w) < 1$, then $0 < \alpha^* < 1$,*
3. *if $\bar{\phi} > 1$, then $\alpha^* = 0$.*

The REE is stable if and only $\alpha < \alpha^*$, i.e., the proportion of informed agents is low enough: Information revealed to some uninformed agents can only destabilize the REE. An intuition in line with Proposition 1 stems from the sensitivity of individual forecasts to others' behavior. When an uninformed agent expects the aggregate price forecast to change in some state, the adjustment in his own price forecast will be weighted by the probability of that state occurring. For this reason, his forecasting behavior is less sensitive to others' forecasts than the behavior of an informed agent. The uninformed agent's behavior is consequently easier to predict, which favors stability.

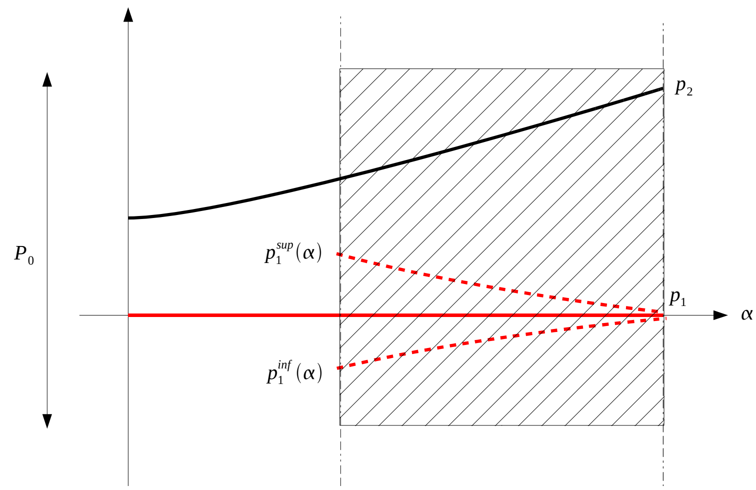

Figure 1: Information structure and rationalizable prices in a two-state case

4.2 A two-state illustration

Figure 1 summarizes our results in a two-state case, with $\phi(1) < 1 < \phi(2)$. The equilibrium price $p^*(1)$ is normalized to 0 in state 1. The black increasing curve depicts the relation between the equilibrium price $p^*(2)$ in state 2 and the proportion α.

By Proposition 1, when all the agents are informed ($\alpha = 1$), the price $p^*(1)$ is the only rationalizable price in state 1 while all prices in P_0 are rationalizable in state 2. By Proposition 3, the equilibrium $(p^*(1), p^*(2))$ is unstable for α large enough ($\alpha > \alpha^*$). Introducing a few uninformed agents thus implies multiple rationalizable prices in state 1, i.e., $p^\infty_{\text{inf}}(1) < p_1 < p^\infty_{\text{sup}}(1)$. However, when α is close to 1, the influence of uninformed agents on the actual outcome can be neglected, so that $p^\infty_{\text{inf}}(1)$ and $p^\infty_{\text{sup}}(1)$ are close to the equilibrium price $p^*(1)$.

It is supposed in the figure that $0 < \alpha^* < 1$ (Corollary 1). By Proposition 2 the set of rationalizable prices in state 2 is P_0 for α high enough. In the figure this set remains equal to P_0 for all $\alpha > \alpha^*$, i.e., both $p^\infty_{\text{inf}}(2) = p^0_{\text{inf}}$ and $p^\infty_{\text{sup}}(2) = p^0_{\text{sup}}$ for all $\alpha > \alpha^*$. The red dashed lines $p^\infty_{\text{inf}}(1)$ and $p^\infty_{\text{sup}}(1)$ represent the boundaries of the set of rationalizable prices in state 1 for $\alpha > \alpha^*$. There is a discontinuity in the set of rationalizable prices in both states when α passes below the threshold α^*: for α just above α^*, uninformed agents expect any price in P_0 to arise in state 2, and for all $\alpha < \alpha^*$ the rationalizable prices reduce to the equilibrium prices in every state.

5 Sunspots and stability

We show how instability results extend to the issue of existence of sunspot equilibria. It is known that existence of sunspot equilibria is closely related to multiplicity of rationalizable outcomes (Guesnerie, 1993). We introduce a concept of sunspot equilibria adapted to our static framework (equilibrium with imperfectly observed sunspots) and we show that the equivalence result holds. Hence, all our (in)stability results apply to the sunspot equilibria: Instability of the equilibrium is associated with non fundamental equilibrium volatility.

Consider a stochastic sunspot variable that can take Σ values ($S = 1, \ldots, \Sigma$), not correlated with fundamentals. Assume that its actual value is not known when agents form their forecasts. Every agent i observes a private signal $s_i = 1, \ldots, \Sigma$ imperfectly correlated with S. Conditionally based on S, private signals are independently and identically distributed across agents, and the probability $\Pr(s_i \mid S)$ that i observes s_i in sunspot event S is independent of i. Thus, in sunspot event S, there are $\Pr(s \mid S)$ agents who observe the signal s ($s = 1, \ldots, \Sigma$).

Suppose that all the agents expect the price $p^e(\omega, S)$ to arise if the state of fundamentals is ω and the sunspot is S. In state (ω, S), there are $\alpha \Pr(s \mid S)$ informed agents whose price forecast is

$$\sum_{S'=1}^{\Sigma} \Pr(S' \mid s) p^e(\omega, S')$$

for any s. There are also $(1-\alpha) \Pr(s \mid S)$ uninformed agents who expect

$$\sum_{S'=1}^{\Sigma} \Pr(S' \mid s) \sum_{w=1}^{\Omega} \pi(w) \, p^e(w, S').$$

Let

$$\mu(S'|S) = \sum_{s=1}^{\Sigma} \Pr(s \mid S) \Pr(S' \mid s)$$

be the average probability (across agents) of sunspot S' if the actual sunspot is S. The aggregate price forecast $P^e(\omega, S)$ is expressed as

$$\sum_{S'=1}^{\Sigma} \mu(S'|S) \left[\alpha p^e(\omega, S') + (1-\alpha) \sum_{w=1}^{\Omega} \pi(w) \, p^e(w, S') \right], \tag{7}$$

and the actual price $p(\omega, S)$, determined by (1) in state (ω, S), is such that

$$p(\omega, S) = \phi(\omega) \, P^e(\omega, S) + \eta(\omega). \tag{8}$$

A REE is a vector of $\Omega\Sigma$ prices $(p^*(1,1), \ldots, p^*(\Omega, \Sigma))$ such that $p^e(\omega, S) = p(\omega, S) = p^*(\omega, S)$ for every (ω, S) in (7) and (8). The 'fundamental' REE is obtained when $p^*(\omega, S)$ is independent of S. Otherwise, sunspots matter and the REE is a 'sunspot' equilibrium.

The following result gives conditions for the existence of a sunspot REE.

Proposition 4. *There exists a sunspot REE if and only if the fundamental REE is unstable in (5).*

In our linear setup, the stability of the fundamental REE is still ruled by Proposition 3 and Corollary 1. Hence, both results also give necessary and sufficient conditions needed for the sunspot REE to exist. In particular, sunspot equilibria exist when many agents are informed about the true state of nature (i.e., $\alpha \geq \alpha^*$).

6 Conclusion

This paper emphasizes the difficulty of coordinating expectations when the sensitivity of the market outcome to agents' forecasts is not common knowledge. A low value of the true sensitivity is not enough for stability. Instead, either a low average sensitivity (when there are many uninformed agents), or even a low sensitivity in every possible state (when there are many informed agents) are needed.

The intuition sustaining these results is easily illustrated in the case with two possible states of nature, one with a low sensitivity and the other with a high sensitivity. Under complete information, the rational expectations equilibrium is stable in the 'low' state, and unstable in the 'high' one. Under asymmetric information, stability properties of the prices in the two states are no longer disconnected. When many agents are informed, multiplicity of rationalizable prices arises in the high state (as in the complete information case). Uninformed agents then fail to predict a unique price in the high state. By contagion this failure implies multiple rationalizable prices in the low state.

These results may possibly contribute to the debate about the transparency of economic policy. They suggest that the disclosure of information about parameters which influence the sensitivity of the economy to agents' beliefs, e.g., the slope of the aggregate demand function in the Muth setup, may be harmful to stability. A government agency or a central bank revealing that the underlying sensitivity is low may destabilize the equilibrium if it cannot convince all the agents to believe its announcement: Instability occurs between full ignorance and full common knowledge.

References

[1] Angeletos G, Pavan A (2004) Transparency of information and coordination in economies with investment complementarities. Am Econ Rev 94: 91–98

[2] Angeletos G, Pavan A (2007) Efficient use of information and social value of information. Econometrica 75: 1103–1142

[3] Arrondel L, Calvo-Pardo H, Tas D (2012) Subjective Return Expectations, Information and Stock Market Participation: Evidence from France. mimeo

[4] Bapat R, Raghavan T (1997) Nonnegative Matrices And Applications. Cambridge University Press, Cambridge

[5] Bernheim B (1984) Rationalizable Strategic Behavior. Econometrica 52: 1007–1028

[6] Cornand C, Heinemann F (2010) Measuring agents' reaction to private and public information in games with strategic complementarities. CESifo working paper 2947

[7] Dominitz J, Manski C (2007) Expected Equity Returns and Portfolio Choice: Evidence from the Health and Retirement Study. J European Econ Association 5: 369–79

[8] Dominitz J, Manski C (2011) Measuring and Interpreting Expectations of Equity Returns. J Appl Econometrics 26: 352–370

[9] Guesnerie R (1992) An exploration of the eductive justification of the rational expectations hypothesis. Am Econ Rev 82: 1254–1278

[10] Guesnerie R (1993) Successes and failures in coordinating expectations. European Econ Rev 37: 243–268

[11] Guesnerie R, Jara-Moroni P (2011) Expectational coordination in simple economic contexts. Econ Theory 47: 205–246

[12] Hellwig C (2005) Heterogeneous information and the welfare effects of public information disclosures. UCLA Economics Online Papers 283

[13] Morris S, Shin H (2002) The social value of public information. Am Econ Rev 92: 1521–1534

[14] Pearce D (1984) Rationalizable Strategic Behavior and the Problem of Perfection. Econometrica 52: 1029–1050

A Proof of Proposition 2

1. Since the equilibrium is unstable, there is a state ω such that $\{p^*(\omega)\} \subsetneq P^\infty(\omega)$. The set of rationalizable price forecasts of uninformed agents cannot be reduced to a single element. In any given state the set of rationalizable prices is determined by the aggregate price forecast in that state, which depends on the forecasts of uninformed agents. Thus in any given state the aggregate price forecast cannot reduce to a single point.

2. Consider the minimum rationalizable prices $\left(p^\infty_{\inf}(\omega)\right)_\omega$. For every ω, $P^\infty(\omega) \subsetneq P^0$. In the case where $p^\infty_{\inf}(\omega) > p^0_{\inf}$ for every ω, the definition of $P^\infty(\omega)$ implies $P^\infty(\omega) = \phi(\omega) P^\infty(\omega) + \eta(\omega)$, and thus, for every ω:

$$p^\infty_{\inf}(\omega) = \alpha\phi(\omega)\, p^\infty_{\inf}(\omega) + (1-\alpha)\,\phi(\omega) \sum \pi(w)\, p^\infty_{\inf}(w) + \eta(\omega).$$

Since the equilibrium price $p^*(\omega)$ is the unique solution of this equation, $p^\infty_{\inf}(\omega) = p^*(\omega)$ for every ω. The same holds true for the maximum rationalizable prices. This shows the first statement.

For ω such that $\alpha\phi(\omega) > 1$, we show that $p^\infty_{\inf}(\omega) = p^0_{\inf}$. To this purpose, we show that, when everyone expects $\left(p^\infty_{\inf}(\omega)\right)_\omega$, we have

$$p^0_{\inf} \geq \alpha\phi(\omega)\, p^0_{\inf} + (1-\alpha)\,\phi(\omega) \sum \pi(w)\, p^\infty_{\inf}(w) + \eta(\omega), \tag{9}$$

which means that $p^0_{\inf}$ is the actual price in state ω (that is: $p^0_{\inf} = p^\infty_{\inf}(\omega)$). Recall the fixed point relation characterizing the equilibrium $\left(p^*(\omega)\right)_\omega$

$$p^*(\omega) = \alpha\phi(\omega)\, p^*(\omega) + (1-\alpha)\,\phi(\omega) \sum \pi(w)\, p^*(w) + \eta(\omega).$$

Subtracting this equality to (9) gives

$$\Delta p(\omega) > \alpha\phi(\omega)\, \Delta p(\omega) + (1-\alpha)\,\phi(\omega) \sum \pi(w)\, \Delta p(w),$$

where $\Delta p(\omega) = p^\infty_{\inf}(\omega) - p^*(\omega) \leq 0$. This rewrites

$$(1-\alpha\phi(\omega))\, \Delta p(\omega) \geq (1-\alpha)\,\phi(\omega) \sum \pi(w)\, \Delta p(w),$$

which holds true as

$$(1-\alpha\phi(\omega))\, \Delta p(\omega) \geq 0 \geq (1-\alpha)\,\phi(\omega) \sum \pi(w)\, \Delta p(w).$$

The same argument shows that $p^\infty_{\sup}(\omega) = p^0_{\sup}$ for every ω such that $\alpha\phi(\omega) > 1$. This shows the second statement.

3. The third item follows from the first step of the iterative process. By assumption, the equilibrium price $\eta(\omega)/(1-\phi(\omega))$ under complete information belongs to P^0. From (5), at the first step of the process, we have:

$$p^1_{\inf}(\omega) = \max\left(p^0_{\inf}, \phi(\omega)\, p^0_{\inf} + \eta(\omega)\right).$$

   Since $p^0_{\inf} < \eta(\omega)/(1-\phi(\omega)$ and $\phi(\omega) < 1$, we have $p^1_{\inf}(\omega) > p^0_{\inf}$. By definition, the map $R_\omega(P)$ cannot be increasing with τ. It follows that $p^\infty_{\inf}(\omega) \geq p^1_{\inf}(\omega) > p^0_{\inf}$. The same argument shows that

$$p^1_{\sup}(\omega) = \min\left(p^0_{\sup}, \phi(\omega)\, p^0_{\sup} + \eta(\omega)\right) < p^0_{\sup},$$

   so that $p^\infty_{\sup}(\omega) \leq p^1_{\sup}(\omega) < p^0_{\sup}$. This shows that $P^\infty(\omega)$ is a strict subset of P^0.

B Proof of Proposition 3

Consider, e.g., the Ω equations in (5) corresponding to the lowest bounds $P^\tau_{\inf}(\omega)$ of $P^\tau(\omega)$. They can be rewritten in matrix form $\mathbf{p}^{\tau+1}_{\inf} = \mathbf{M}\mathbf{p}^\tau_{\inf} + \eta$, where $\mathbf{p}^\tau_{\inf}$ is the $\Omega \times 1$ vector $(P^\tau_{\inf}(1), \ldots, P^\tau_{\inf}(\Omega))$, η is the $\Omega \times 1$ vector $(\eta(1), ..., \eta(\Omega))$, and $\mathbf{M}$ is the $\Omega \times \Omega$ matrix $\alpha\mathbf{\Phi} + (1-\alpha)\,\mathbf{\Phi\Pi}$ (with $\mathbf{\Phi}$ the diagonal $\Omega \times \Omega$ matrix whose $\omega\omega$th entry is $\phi(\omega)$, and $\mathbf{\Pi}$ the $\Omega \times \Omega$ stochastic matrix whose $\omega\omega'$th entry is $\pi(\omega')$). The REE is stable if and only if the spectral radius $\rho(\mathbf{M})$ of $\mathbf{M}$ is less than 1. The proof now hinges on the fact that for any $\Omega \times \Omega$ positive matrix $\mathbf{M}$, and any $\Omega \times 1$ vector $\mathbf{x} = (x_\omega)$ with every $x_\omega > 0$, we have

$$\min_\omega \frac{(\mathbf{Mx})_\omega}{x_\omega} \leq \rho(\mathbf{M}) \leq \max_\omega \frac{(\mathbf{Mx})_\omega}{x_\omega},$$

where $(\mathbf{Mx})_\omega$ stands for the ωth component of the $\Omega \times 1$ vector $\mathbf{Mx}$ (see Lemma 3.1.2. in Bapat and Raghavan(1997)). Let

$$Q(\mathbf{x}, \omega) = \frac{(\mathbf{Mx})_\omega}{x_\omega} = \phi(\omega)\left[\alpha + (1-\alpha)\frac{1}{x_\omega}\sum_{w=1}^{\Omega}\pi(w)\, x_w\right],$$

for any ω. Assume first that $\alpha\phi(\omega) > 1$ for some ω, e.g. $\omega = \Omega$. Then, consider the vector $\mathbf{x} = (\varepsilon, \ldots, \varepsilon, 1)'$ where $\varepsilon > 0$. When ε tends toward 0, $Q(\mathbf{x}, \omega)$ tends to $(+\infty)$ for every $\omega < \Omega$, and $Q(\mathbf{x}, \Omega) \geq \alpha\phi(\Omega) > 1$. Hence, $\min_\omega Q(\mathbf{x}, \omega) > 1$ for ε small enough, and so $\rho(\mathbf{M}) > 1$: The REE is unstable if $\alpha\phi(\omega) > 1$ for some ω. If, on the contrary, $\alpha\phi(\omega) < 1$ for any ω, then define

$$E = \sum_{w=1}^{\Omega}\pi(w)\frac{(1-\alpha)\,\phi(w)}{1-\alpha\phi(w)}.$$

Consider the $\Omega \times 1$ positive vector $\mathbf{x}$ whose ωth component is

$$x_\omega = \frac{1}{E}\frac{(1-\alpha)\,\phi(\omega)}{1-\alpha\phi(\omega)}.$$

If $E \geq 1$, then $Q(\mathbf{x}, \omega) > 1$ for any ω, so that $\min_\omega Q(\mathbf{x}, \omega) \geq 1$, and the REE is unstable. If, on the contrary, $E < 1$, then $Q(\mathbf{x}, \omega) < 1$ for any ω, so that $\max_\omega Q(\mathbf{x}, \omega) < 1$, and the REE is stable.

C Proof of Corollary 1

1. Assume first that $\phi(\omega) < 1$ for any $\omega = 1, \dots, \Omega$. Then, $\alpha\phi(\omega) < 1$ and $(1-\alpha)\phi(\omega) / (1 - \alpha\phi(\omega)) < 1$ for any ω. By Proposition 3, the REE is stable.

2. Let now $\inf_\omega \phi(\omega) < 1 < \sup_\omega \phi(\omega)$. If $\alpha > 1/\sup_\omega \phi(\omega)$, the REE is unstable, by Proposition 3. If $\alpha \leq 1/\sup_\omega \phi(\omega)$, then $\alpha\phi(\omega) < 1$ for every ω, and the REE is stable if and only if (6) is met. Let

$$F(\alpha) = \sum_{w=1}^{\Omega} \pi(w) \frac{\phi(w)}{1 - \alpha\phi(w)} - \frac{1}{(1-\alpha)} \tag{10}$$

Since $F(\cdot)$ is a continuous and increasing function of α on the interval $[0, 1/\sup_\omega \phi(\omega)]$, with $F'(\alpha) > 0$ whatever α is, there is at most one value α such that $F(\alpha) = 0$ on this interval. Observe now that $F(0) = \bar{\phi} - 1$, and $F(\alpha)$ tends to $+\infty$ when α tends to $1/\sup_\omega \phi(\omega)$ from below. If, on the one hand, $\bar{\phi} \geq 1$, then $F(\alpha) \geq F(0) > 0$ for any $\alpha \in [0, 1/\sup_\omega \phi(\omega)]$, and the stability condition (6) is never satisfied. If, on the other hand, $\bar{\phi} < 1$, then there exists a unique solution α^* ($\alpha^* > 0$) to $F(\alpha) = 0$ in $[0, 1/\sup_\omega \phi(\omega)]$. The condition $F(\alpha) < 0$, i.e. the stability condition (6), is equivalent to $\alpha < \alpha^*$. Since $F(\alpha^*) = 0$ implicitly defines α^* as a function $(\phi(1), \dots, \phi(\Omega))$, and since $F(\cdot)$ increases in every $\phi(\omega)$, α^* decreases in every $\phi(\omega)$.

3. Let $\bar{\phi} > 1$. We know that $F(\alpha) > 0$ for any $\alpha \in [0, 1/\sup_\omega \phi(\omega)]$. As a result, the stability condition (6) is never satisfied.

D Proof of Proposition 4

Let us rewrite conditions (8) in matrix form. To this aim, let $\mathbf{p}(S)$ be the $\Omega \times 1$ vector whose ωth component is $p(\omega, S)$, and $\mathbf{p}$ be the $\Omega\Sigma \times 1$ vector $(\mathbf{p}(1), \dots, \mathbf{p}(\Sigma))$. Let $\mathbf{S}$ be the $\Sigma \times \Sigma$ stochastic matrix whose $S'S$th entry is $\mu(S'|S)$. Then, with $\mathbf{M}$ defined in Proposition 3, a REE is a vector $\mathbf{p}$ such that

$$\mathbf{p} = (\mathbf{M} \otimes \mathbf{S})\,\mathbf{p} + \mathbf{1}_\Sigma \otimes \eta, \tag{11}$$

where the symbol $\otimes$ stands for the Kronecker product. Let $e(S)$ be the Sth eigenvalue of $\mathbf{S}$, with $e(S) \in [-1, 1]$ since $\mathbf{S}$ is a stochastic matrix. Let $\mu(\omega)$ be the ωth eigenvalue of $\mathbf{M}$. Then, the $\Omega\Sigma$ eigenvalues of $\mathbf{M} \otimes \mathbf{S}$ are $e(S)\mu(\omega)$ for any pair (ω, S). If $\rho(\mathbf{M}) < 1$, then all the eigenvalues of $\mathbf{M} \otimes \mathbf{S}$ have moduli less than 1, and so $\mathbf{M} \otimes \mathbf{S} - \mathbf{I}_{2\Omega}$ is invertible and there is a unique REE. If $\rho(\mathbf{M}) \geq 1$, there exist stochastic matrices such that $e(S) = 1/\rho(\mathbf{M})$ for some S. In this case, the matrix $\mathbf{M} \otimes \mathbf{S}$ has an eigenvalue equal to 1, and there are infinitely many $\mathbf{p}$ solution to (11), i.e. infinitely many sunspot REE and the fundamental REE.